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Nice Bases of QTAG-Modules

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ABSTRACT

A module M over an associative ring R with unity is a QTAG-module if every finitely generated submodule of any homomorphic image of M is a direct sum of universal modules. In this paper, we investigate the class of QTAG-modules having nice basis. It is proved that if H_ω (M) is bounded then M has a bounded nice basis and if H_ω (M) is a direct sum of uniserial modules, then M has a nice basis. We also proved that if M is any QTAG-module, then M⨁D has a nice basis, where D is the h-divisible hull of H_ω (M).

KEYWORDS
QTAG-module, Bounded submodule, Separable submodules, nice submodules

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1. Introduction

Many concepts for group like purity, projectivity, injectivity, height etc. have been generalized for modules. To obtain results of groups which are not true for modules either conditions have been applied on modules or upon the underlying rings. We imposed the condition on modules that every finitely generated submodule of any homomorphic image of the module is a direct sum of uniserial modules while the rings are associative with unity. After these conditions many elegant results of groups can be proved for $QTAG$-modules which are not true in general. Many results of this paper are the generalization of the papers by Danchev (2007) and Danchev and Keef (2011a).

The study of $QTAG$-modules was initiated by Singh (1979). Mehdi (1984), Mehdi (1985), and Mehdi *et al.* (2005) worked a lot on these modules. They studied different notions and structures of $QTAG$-modules and developed of theory of these modules by introducing several notions, investigated some interesting properties and characterized them. Yet there is much to explore.

In this paper, we shall deal with the modules with a nice basis i.e., the modules which have special representation in terms of certain nice submodules. This class of modules is very large. Our main aim of this paper is to generalize the concept of nice basis of $QTAG$-modules and discuss several results of modules having nice basis. For the detailed literature on nice basis of groups one can go through the following papers: Danchev (2005), Danchev (2007), Danchev and Keef (2011a) and Danchev and Keef (2011b).

2. Preliminaries

All the rings R considered here are associative with unity and right modules M are unital $QTAG$-modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = sup\left\{d\left(\frac{yR}{xR}\right) \mid y \in M, x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k (Khan, 1978). $H_\omega(M)$ denotes the first Ulm-submodule of a module M consisting of all elements of infinite height and $H_{\omega+n}(M) = H_n(H_\omega(M))$. M is the h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. A submodule N ⊆ M may not be h-divisible in M but if it is contained in some h-divisible submodule K ⊆ M, then it is said to be semi h-divisible and the minimal h-divisible submodule K of M containing N is said to be the h-divisible hull of N.

A $QTAG$-module M is said to be separable, if $M^1 = 0$. A module M is said to be bounded, if there exists an integer n such that $H(x) \leq n$ for every uniform element $x \in M$ (Singh,1979). A submodule N of a $QTAG$-module M is a nice, if for every ordinal σ, there exists an element $x_\sigma \in N_{\sigma+1}/N_\sigma$ which is proper with respect to N_σ (Mehdi *et al.*, 2006). A family of nice submodules $\mathcal{N}$ of submodules of M is called nice system in M if

(i) $0 \in \mathcal{N}$;
(ii) If $\{N_i\}_{i \in j}$ is any subset of $\mathcal{N}$, then $\sum_i N_i \in \mathcal{N}$;
(iii) Given any $N \in \mathcal{N}$ and any countable subset X of M, there exist $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated.

A h-reduced $QTAG$-module M is called totally projective if it has a nice system. Let M be a QTAG- module. It defines a well ordered sequence of submodules $M = M^0 \supset M^1 \supset M^2 \supset \ldots \supset M^\tau = 0$, for some ordinal τ. Here $M^1 = \bigcup_{k \in \omega} H_k(M)$, $M^{\sigma+1} = (M^\sigma)^1$ and $M^\sigma = \bigcap_{\rho < \sigma} M^\rho$, if σ is a limit ordinal. M^σ is said to be the σ-th Ulm submodule of M.

Several results which hold for TAG-modules also hold good for $QTAG$-modules (Singh, 1979). Notations and terminology are follows from (Fuchs, 1970 and 1973).

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3. Main Results

The concept of nice basis for the groups was introduced by Danchev (2005) using nice subgroups. Motivated by this, we generalize this concept to nice basis of $QTAG$-modules as follows:

Definition 3.1: A $QTAG$-modules M has a nice basis if it can be expressed as $M = \cup_{k<\omega} M_k$, $M_k \subseteq M_{k+1} \subseteq M$ and each M_k is nice in M and a direct sum of uniserial modules.

Remark 3.2: If each M_k is bounded then M has a bounded nice basis.

Let us start with a useful result:

Propositions 3.3: Let N be a submodule of M such that $H_\omega(N) = H_\omega(M)$. If M has a (bounded) nice basis, then N also has a (bounded) nice basis.

Proof: Let $\{M_k\}_{k<\omega}$ be the (bounded) nice basis of M. Now $M = \cup_{k<\omega} M_k$, $M_k \subseteq M_{k+1}$ and every M_k is a (bounded) nice basis of M, which is a direct sum of uniserial modules. Now $N = \cup_{k<\omega} (M_k \cap N)$ and all the intersections are (bounded) and the direct sums of uniserial modules. For a limit ordinal σ,

$$\cap_{\rho<\sigma}\left((M_k \cap N) + H_\rho(N)\right) \subseteq \cap_{\rho<\sigma}\left(M_k + H_\rho(M)\right) \cap N = (M_k + H_\sigma(M)) \cap N = (M_k + H_\sigma(N) \cap N = H_\sigma(N + (M_k \cap N)$$

and the result follows:

Corollary 3.4: A direct summand of a module M with a (bounded) nice basis and a separable complement also has a (bounded) nice basis.

Proof: Let N be a direct summand of M, with a separable complement K. Now $M = K \oplus N$. Since $H_\omega(K) = 0, H_\omega(M) = H_\omega(N)$ and by Proposition 3.3, N has a (bounded) nice basis.

Propositions 3.5: Let N be nice submodule of M such that M/N has a bounded nice basis. Then

(i) If N is bounded, then M has a bounded nice basis;
(ii) If N is a direct sum of uniserial modules, then M has a nice basis.

Proof: We may express $M/N = \cup_{k<\omega} (M_k/N)$ where $M_k \subseteq M_{k+1} \subseteq M, M_k/N$ is nice in M/N and it is bounded. Now by (Mehdi, *et al.*, 2005) M_k is nice in M. Since N is bounded M_k must be bounded and (i) follows.

Again M_k/N is bounded by (Singh, 1979), M_k is a direct sum of uniserial modules. Now, $M = \cup_{k<\omega} M_k$ and (ii) follows:

Remark 3.6: Since $H_\sigma(M)$ is nice in M for every ordinal σ, if $M/H_\sigma(M)$ has a bounded nice basis and $H_\sigma(M)$ is a direct sum of uniserial modules, then M has a nice basis.

Remark 3.7: If $M/H_\sigma(M)$ has a bounded nice basis and $H_\sigma(M)$ is bounded then M has a bounded nice basis. Here σ is any ordinal number. Also, if M has a bounded nice basis, then $H_\sigma(M)$ and $M/H_\sigma(M)$ have bounded nice bases.

Remark 3.8 If the length of M $\alpha < \omega.2$, i.e., $\alpha = \beta + \omega$ for some ordinal β and $M/H_\beta(M)$ has a bounded nice basis, then M has a bounded nice basis.

Proposition 3.9: If $H_\omega(M)$ is a direct sum of uniserial modules, then M has a nice basis. If $H_\omega(M)$ is bounded too then M has a bounded nice basis.

Proof: Since $M/H_\omega(M)$ is separable, it has a bounded nice basis by Remark 3.8. We may express $M/H_\omega(M)$ as $\cup_{k<\omega} H^k (M/H_\omega(M))$. Again $H^k(M/H_\omega(M))$= $M_k/H_\omega(M)$ for some modules M_k such that $M_k \subseteq M_{k+1} \subseteq M$ and $H_k(M_k) \subseteq H_\omega(M)$. Thus $M = \cup_{k<\omega} M_k$ and $H_k(M_k)$ is a direct sum of uniserial modules, M_k is also a direct sum of uniserial modules (Singh, 1979). Since $M_k/H_\omega(M)$= $H^k(M/H_\omega(M))$ is nice in $M/H_\omega(M)$ and $H_\omega(M)$ is nice in M by (Mehdi, *et al.* 2005) every M_k is nice in M and the result follows.

Proposition 3.10: Let α be an ordinal such that $M/H_\alpha(M)$ is countably generated and $H_\alpha(M)$ has a (bounded) nice basis. Then M has a (bounded) nice basis.

Proof: Let $M/H_\alpha(M) = \cup_{k<\omega} H^k (M/H_\alpha(M))$ where $M_k \subseteq M_{k+1} \subseteq M$ and $M_k/H_\alpha(M)$ are finitely generated for every $k \in Z^+$. Now $M = \cup_{k<\omega} M_k$ and for every k, $M_k = H_\alpha(M) + T_k$ where T_k are finitely generated and $T_k \subseteq T_{k+1}$. Again

$$H_\alpha(M) = \cup_{k<\omega} N_k \quad ,$$

$N_k \subseteq N_{k+1} \subseteq H_\alpha(M)$, such that N_k are nice in $H_\alpha(M)$) and M and N_k are (bounded) direct sums of uniserial modules. Now $M = \cup_{k<\omega}(N_n + T_n)$ where $N_n + T_n$ are nice in M, and are (bounded) direct sums of uniserial modules (Singh, 1979) and the result follows.

The following result is an immediate consequence of Proposition 3.10:

Corollary 3.11: If the length of the module M is less than $\omega.2$ and $M/H_\omega(M)$ is countably generated then M has a bounded nice basis.

Proposition 3.12: If M is a module such that $H_\omega(M)$ is countably generated, then M is the union of a countable ascending chain tower of nice direct sums of countably generated modules.

Proof: Since the separable modules have a bounded nice basis $M/H_\omega(M) = \cup_{k<\omega}(M_k/H_\omega(M))$ where $M_k/H_\omega(M) \subseteq M_{k+1}/H_\omega(M)$ are nice submodules of $M/H_\omega(M)$ and they are bounded such that $H_k(M_k) \subseteq H_\omega(M)$. Now M_k's are nice in M and they are direct sums of countably generated modules and $M = \cup_{k<\omega} M_k$, the result follows.

Following is the immediate consequence of the above propositions:

Corollary 3.13: If M is a $QTAG$-module of length at most $\omega.2$ such that $H_\omega(M)$ is countably generated, then M has a nice basis.

Proposition 3.14: Direct sums of modules with a bounded nice basis have bounded nice basis.

Proof: Let M be a $QTAG$-module such that $M = \underset{i\in I}{\oplus} M_i$, where each M_i has a bounded nice basis. Let $M_i = \cup_{j<\omega}\{M_{ij}\}$, where $M_{ij} \subseteq M_{ij+1}$, all M_{ij} are bounded by j and M_{ij}'s are nice in M_i. Therefore,

$$M = \underset{i\in I}{\oplus}\left(\bigcup_{j<\omega} M_{ij}\right) = \bigcup_{j<\omega}(\oplus M_{ij}) = \bigcup_{j<\omega} P_j$$

where each $P_j = \underset{i\in I}{\oplus} M_{ij}$. Now $P_j \subseteq P_{j+1} \subseteq M$, $H^j(P_j) = 0$ and P_j is nice in M.

Now, we will see the behaviour of a $QTAG$-module which contains a submodule with a nice basis under the action of a countable extension of quotients.

Proposition 3.15: Suppose $M = N$ is countably generated, where N is a balanced submodule of a $QTAG$-module M. If N has a (bounded) nice basis, then M has (bounded) nice basis.

Proof: Since we can write $M \cong (N + M)/N$, (Fuchs,1970) the proof follows from Proposition 3.14.

A fully invariant submodule L of M is large in M if $L + B = M$, for every basic submodule B of M (Sikander *et al.,* 2017). Now, we have the following result:

Corollary 3.16: Let L be any large submodule of a $QTAG$-module M. Then L has a (bounded) nice basis if M has a (bounded) nice basis.

Proof: Since $H_\omega(M) = H_\omega(L)$ (Sikander *et al.,* 2017)., the result follows immediately in view of Proposition 3.3.

We will try to explore the possibility that whether to have a nice basis can be invariantly retrieved under the action of bounded quotients.

Theorem 3.17: Let M be a $QTAG$-module with a submodule N such that $M = N$ is bounded. Then M has a bounded nice basis if and only if N has a bounded nice basis.

Proof: Since $H_m(M) \subseteq N$ for some $m \in Z^+, H_\omega(M) = H_\omega(N)$ and by Proposition 3.1, N has a bounded nice basis. For the converse, suppose N has a bounded nice basis $\{P_n\}_{n<\omega}$. Now $N = \bigcup_{n<\omega} P_n$, $P_n \subseteq P_{n+1}$ and for every $n \geq 1$, P_n is nice in N which is a direct sum of uniserial modules. Therefore, $H_m(M) = \bigcup_{n<\omega}(P_n \cap H_m(M))$, where $P_n \cap H_m(M) \subseteq P_{n+1} \cap H_m(M)$. Since $P_n \cap H_m(M)$ are submodules $P_n{}'s$ of they are bounded and the direct sum of uniserial modules.

On the other hand, for every limit ordinal σ ,

$$\bigcap_{\rho<\sigma}(P_n + H_\rho(M)) = \bigcap_{\rho\leq\rho<\sigma}(P_n + H_\rho(M))$$
$$\subseteq \bigcap_{\rho<\sigma}(P_n + H_\rho(M)) = P_n + H_\sigma(N)$$
$$= P_n + H_\sigma(M)$$

Since P_n are nice in M; all $P_n \cap H_m(M)$ are nice in $H_m(M)$: We infer that $H_m(M)$ has a bounded nice basis, therefore by Proposition 3.9, M has a bounded nice basis as required.

For any $QTAG$-module M, $g(M)$ denotes the smallest cardinal number λ such that M admits a generating set X of uniform elements of cardinality λ i.e., $\#(X) = \lambda$ (Mehran and Singh, 1986). Any $QTAG$-module $M = K \oplus D$, where D is h-divisible and K is h-reduced. M has a nice basis if the h-reduced part K has a nice basis. Here we show that if M is a $QTAG$-module and D is a h-divisible module such that $g(D) \geq g(H_\omega(M))$. then $M \oplus D$ must have a nice basis. We define $Bg(M)$ as the cardinality of the minimal generating set of a h-divisible module D such that $M \oplus D$ has a nice basis and we find that $B_g(M) \leq g(H_\omega(M))$.

Theorem 3.18: Let M be a $QTAG$-module and D, the h-divisible hull of $H_\omega(M)$. Then $M \oplus D$ has a nice basis.

Proof: Suppose $M \cong M' \oplus D'$, where M' is h-reduced and D' is h-divisible. If D'' is the h-divisible hull for $H_\omega(M')$, then we may put $D = D' \oplus D''$ and assume that M is h-reduced. Suppose $H_\omega(M)$ is finitely generated. Now consider the inclusion map $f_1: H_\omega(M) \to D$ is injective and the identity map $f_2: H_\omega(M) \to H_\omega(M)$. If $H_\omega(M)$ is infinitely generated, then we may define K as the external direct sum $\sum_{x_i \in H_\omega(M)} x_i R$.

Now $g(K) = g(H_\omega(M)) = g(D)$.

Therefore, we may define the embedding $f_1: K \to D$ and $f_2: K \to H_\omega(M)$. In $M \oplus D$, we put $K' = \{(f_2(x), x \mid x \in K\}$ and $D' = \{(0, y) | y \in D\}$.

Now $(K' + D')/K' = (H_\omega(M) \oplus D)/K'$ is a h-divisible submodule of $(M \oplus D)/K'$ so that

$$\frac{M \oplus D}{K'} \cong \frac{(H_\omega(M) \oplus D)}{K'} \oplus \frac{P}{K'},$$

where P is a submodule of $M \oplus D$ containing K' such that

$$\frac{P}{K'} \cong \frac{(M \oplus D)/K'}{(H_\omega(M) \oplus D)/K'} \cong \frac{M \oplus D}{H_\omega(M) \oplus D)} \cong \frac{M}{H_\omega(M)}.$$

If $n < \omega$, let P_n be the submodule of P containing K' such that $(P_n/K') \cong H^n(M/H_\omega(M))$. We may define $Q_n = P_n + H^n(D')$. We have to show that $\{Q_n\}_{n<\omega}$ forms a nice basis for $M \oplus D$. Since $P_n \subseteq P_{n+1}$ and $H^n(D') \subseteq H^{n+1}(D')$, $Q_n = P_n + H^n(D') \subseteq P_{n+1} + H^{n+1}(D') = Q_{n+1}$ and the first condition is satisfied. Now the map $x \to (f_2(x), x)$ extends to an isomorphism and $K \cong K' = Q_0$. If $n > 0$; then $H_n(P_n) \subseteq K'$ and $H_n(D') = 0$ so that $H_n(Q_n) \subseteq K$. Since K' is a direct sum of uniserial modules so is $H_n(Q_n)$ and Q_n.

It remains to show that $\bigcup Q_n = M \oplus D$ and each Q_n is nice in $M \oplus D$. Using the above mentioned isomorphisms, we conclude that

$$\frac{M \oplus D}{K'} \cong \frac{(H_\omega(M) \oplus D)}{K'} \oplus \frac{M}{H_\omega(M)}.$$

The submodule $\frac{Q_n}{K'}$ is mapped onto $\left(\frac{H^n(D')\oplus K)}{K'}\right) \oplus H^n\left(\frac{M}{H_\omega(M)}\right)$. Since $D' = \bigcup_{n<\omega} H^n(D')$, $H_\omega(M) \oplus D = D' + K'$ and $\frac{M}{H_\omega(M)} = \bigcup_{n<\omega} H^n\left(\frac{M}{H_\omega(M)}\right)$, we have $M \oplus D = \bigcup_{n<\omega} Q_n$. These two equations imply that

$$\frac{M \oplus D}{Q_n} \cong \frac{(M \oplus D)/K'}{Q_n/K'}$$
$$\cong \left(\frac{H_\omega(M) \oplus D}{H^n(D') + K'}\right) \oplus \left(\frac{M/H_\omega(M)}{H^n(M/H_\omega(M))}\right)$$
$$\cong \left(\frac{D' \oplus K'}{H^n(D') + K'}\right) \oplus H_n\left(\frac{M}{H_\omega(M)}\right).$$

Now any element of $\frac{D' \oplus K'}{H^n(D') + K'}$ is represented by an $x \in D'$, which is h-divisible, hence $H(x) = \infty$, so x is proper in this coset. If $x \in H_n\left(\frac{M}{H_\omega(M)}\right)$ such that $H(x)$ is finite, then such a coset always has a proper element. Therefore, Q_n is nice and $\{Q_n\}_{n<\omega}$ is a nice basis.

Following is the immediate consequence of the above theorem:

Corollary 3.19: Any $QTAG$-module M is a direct summand of a $QTAG$-module with a nice basis.

Proposition 3.20: Let M, D be $QTAG$-modules such that D is h-divisible and $g(D)$ is finite. Then $M \oplus D$ has a nice basis if and only if M has a nice basis.

Proof: If M has a nice basis, then $M \oplus D$ has a nice basis. For the converse, suppose $M \oplus D$ has a nice basis $\{P_n\}_{n<\omega}$. We may express $M = M_0 \oplus D_0$, where M_0 is h-reduced and D_0 is h-divisible. If $g(D_0)$ is infinite, then $M \oplus D \cong M_0 \oplus D_0 \oplus D \cong M_0 \oplus D_0 = M$, and the result follows.

Otherwise, if $g(D_0)$ is finite, then we shall prove that M_0 has a nice basis. Replacing M by M_0 and D by $D_0 \oplus D$ we may assume that M is h-reduced. If $D' = \{0\} \oplus D$, then $D' = H_\omega(M \oplus D)$ and $Q_n = (P_n + D')/D'$ is nice in $(M \oplus D)/D' \cong M$. Since $\bigcup_{n<\omega} P_n = M \oplus D$, we have $\bigcup_{n<\omega} Q_n = (M \oplus D)/D' \cong M$.

Now for each $n < \omega$; there exists an isomorphism $P_n/(P_n \cap D') \to (P_n + D')/D' = Q_n$ .Since $g(D')$ is finite, hence $P_n \cap D'$ must be a direct sum of uniserial modules and $P_n \cap D'$ is finitely generated. Therefore, Q_n is a direct sum of uniserial modules and $\{Q_n\}_{n<\omega}$ is a nice basis for M.

Corollary 3.21: If M is a $QTAG$-module, then $Bg(M)$ is either 0 and M has a nice basis or $B_g(M)$ is an infinite ordinal such that $B_g(M) \leq g(H_\omega(M))$, then M does not have a nice basis.

Proof $B_g(M)$ =0 if and only if M has a nice basis. If M does not have a nice basis then by Proposition 3.8, $B_g(M)$ is not finite. Thus, by Theorem 3.18, $B_g(M) \leq g(H_\omega(M))$.

Proposition 3.22: For a $QTAG$-module M; the following holds:

(i) If K is a separable $QTAG$-module, then $B_g(M \oplus K) = B_g(M)$.
(ii) If D is a h-divisible module, then then $B_g(M \oplus D)$ is 0, if $g(D) \geq B_g(M)$ or $B_g(M)$ if $g(D) < B_g(M)$.
(iii) If D is h-divisible such that $g(D)$ is finite, then $B_g(M \oplus D) = B_g(M)$.
(iv) For any ordinal σ, $B_g(H_\sigma(M) \leq B_g(M)$.
(v) If $M/H_\omega(M)$ is a direct sum of uniserial modules, then $B_g(M) = B_g(H_\omega(M))$.
(vi) If $H_\omega(M)$ has a nice basis and $M/H_\omega(M)$ is a direct sum of uniserial modules, then $B_g(M) = B_g(H_\omega(M)) = 0$.

Proof: (i) If D is a h-divisible module then by Proposition 3.6, $M \oplus D$ has a nice basis if and only if $M \oplus K \oplus D$ has a nice basis. Therefore $B_g(M \oplus \mathrm{K}) = B_g(M)$.

(ii) If $g(D) \geq B_g(M)$, then $M \oplus \mathrm{D}$ has a nice basis so that $B_g(M \oplus \mathrm{D}) = 0$. Otherwise if $g(D) < B_g(M)$, then for any h-divisible module $D', M \oplus D \oplus D' \cong M \oplus (D \oplus D')$ has a nice basis if and only if $g(D \oplus D') = g(D) + g(D') \geq B_g(M)$ if and only if $g(D') \geq B_g(M)$, if and only if $M \oplus D'$ has a nice basis and the result follows.

(iii) is a direct consequence of (ii), where $B_g(M)$, is 0 or infinite.

(iv) Let $\{P_n\}_{n<\omega}$ be a nice basis for $M \oplus D$, then $\{P_n \cap H_\sigma(M \oplus D)\}_{n<\omega}$ is a nice basis for $H_\sigma(M \oplus D) = H_\sigma(M) \oplus D$.

(v) We have $B_g(H_\omega(M)) \leq B_g(M)$. On the other hand if D is a h-divisible module such that $g(D) = B_g(M)$, then $H_\omega(M) \oplus D = H_\omega(\mathrm{M} \oplus D)$ has a nice basis. Since $\frac{M \oplus D}{H_\omega(M \oplus D)}$ is a direct sum of uniserial modules and $H_\omega(M \oplus D)$ has a nice basis as $(M \oplus D)$ has a nice basis, so that $B_g(M) \leq B_g(H_\omega(M))$ and we are done.

(vi) If $H_\omega(M)$ is a direct sum of uniserial modules, then M and $H_\omega(M))$ both have nice bases and $B_g(M) = 0 = B_g(H_\omega(M))$.

Proposition 3.23: If M and M' are $QTAG$-modules, then $B_g(M \oplus M') \leq B_g(M) + B_g(M')$.

Proof: Let D and D' be h- divisible modules such that $g(D) = B_g(M), g(D') = B_g(M')$. Then,

$$(M \oplus M') \oplus (D \oplus D') \cong (M \oplus D) \oplus (M' \oplus D').$$

Since $M \oplus D$ and $M' \oplus D'$ have nice basis, their direct sum has a nice basis.

The behavior of bounded nice basis is different from that of nice basis. This can be seen as follows:

Theorem 3.24: Let M be a h-reduced module and D is a h-divisible module. Then M has a bounded nice basis if and only if $M \oplus D$ has a bounded nice basis.

Proof: $\{H^n(D)\}_{n<\omega}$ forms a bounded nice basis for a h-divisible module D. Since direct sums of modules with bounded nice basis, the necessity is done and we may consider that $M \oplus D$ has a bounded nice basis $\{Q_n\}_{n<\omega}$. If $D' = \{0\} \oplus D = H_\omega(M \oplus D)$ and $P_n = (Q_n + D')/D'$, then P_n is nice in $(M \oplus D')/D' \cong M$. Moreover, $P_n = Q_n/(Q_n \cap D')$ must be bounded. Therefore, it is a direct sum of uniserial modules. Since $M \oplus D$ is the union of the ascending chain of modules Q_n. $(M \oplus D')/D' \cong M$ is the union of the of the ascending chain of submodules P_n. Therefore, M has a bounded nice basis.

4. Discussion

We end this article with the following observation:

Suppose M is h-reduced module that does not have a nice basis. If D is a h-divisible module such that $g(D) \geq B_g(M)$, then $M \oplus D$ has a nice basis. On the other hand, if M does not have a bounded nice basis, by Theorem 3.24, $M \oplus D$ does not have a bounded nice basis. This means that there are $QTAG$-modules with a nice basis which do not have bounded nice basis.

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